Inpainting & Visual Interpolation

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Example of Inpainting

(A) Scratch Removal (Bertalmio-Sapiro-Caselles-Ballester, *SIGGRAPH,* 2000)

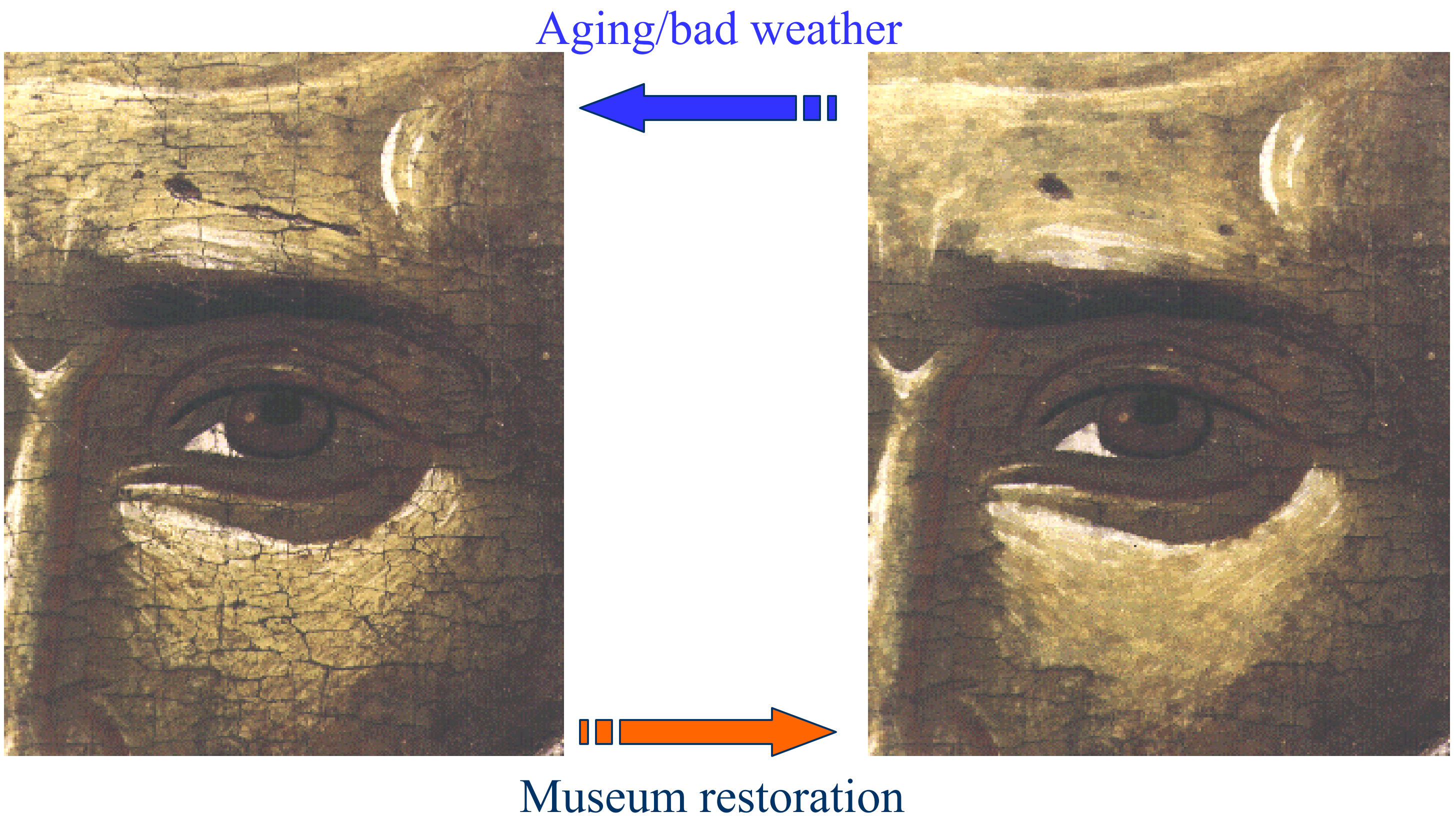
Sir of photoshops:

Could you help me restore this precious photo in 1911?

Nature or you did it

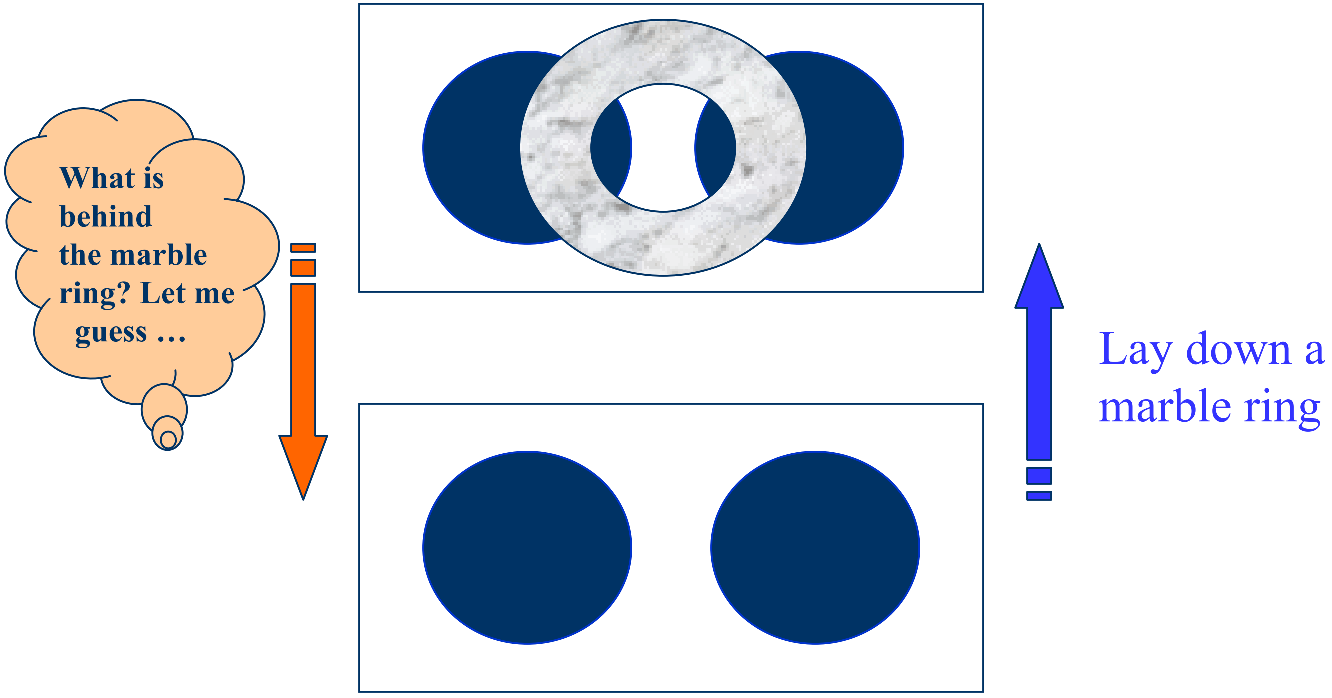
Example of Inpainting

(B) Crack Restoration **(for Digital Museums)** (**Giakoumis-Pitas, 1998**)



Example of Inpainting

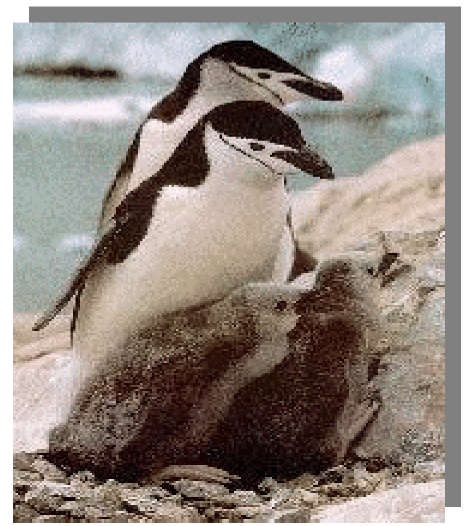
(C) Disocclusion (Nitzberg-Mumford-Shiota, 1993; Masnou-Morel, 1998)



Example of Inpainting

(D) Text Removal (Bertalmio et al., 2000; Chan-Shen, 2001)

adding



removal

Zoom-in (super-resolution, magnification)

Chan-Shen (*SIAM J. Appl. Math.*, 2001), Tsai-Yezzi-Willsky (*IEEE Trans. I. P.*, 2001), Ballester-Bertalmio-Caselles-Sapiro-Verdera (*IEEE Trans. I. P.*, 2001)

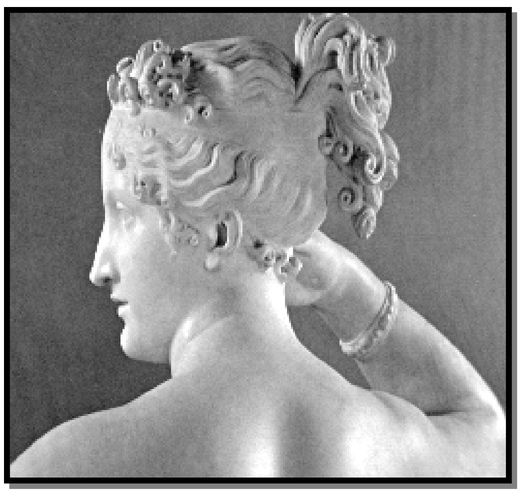
zoom in

zoom out

Image source: Test Image Databank, Computational Vision Group, Caltech.

Primal-Sketch Based Image Coding

Chan-Shen (*SIAM J. Appl. Math.*, 2001)



Edge detection

A primal sketch

Edge decoding.
Is it possible?

David Marr once asked . . .

Image source: Test Image Databank,
Computational Vision Group, Caltech.

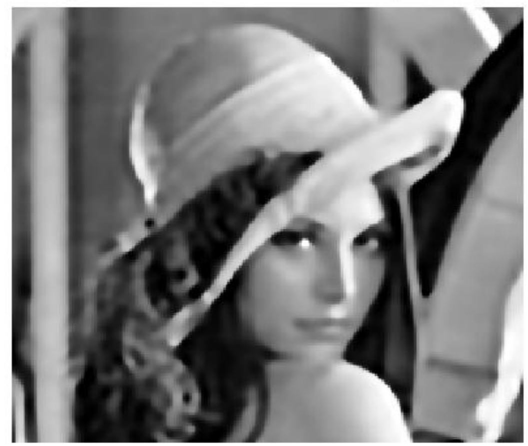
Error Concealment in Wireless Transmission

Chan-Shen [*AMS Contemp. Math.*,2002]

Random packet loss due to transmission

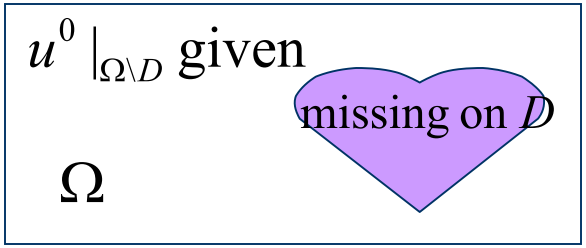
A blurred image with 80 lost packets

Deblurring and error concealment by TV inpainting



Error concealment

What Is Inpainting



- Inpainting = **Image Interpolation**.

  (**initially circulated among museum restoration artists; first introduced into I.P. by Saprio's group [EECS, UMN, 1999])**

- What makes inpainting difficult is the complexity of images**:**
  - **having a large dynamic range of** scales**;**
  - **intrinsically non-smooth due to** edges and boundaries**;**
  - **the missing domains can have** complicated topology**;**

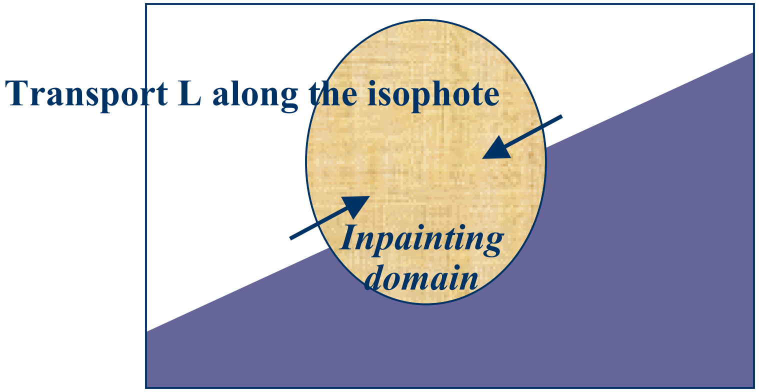
  - **direct classical interpolation tools perform less ideally:**
    - polynomials (Lagrange, Hermite, splines);
    - linear filtering (Fourier, wavelets, linear (heat) diffusion);
    - radially symmetric functions (as in spatial statistics).

3rd Order PDE Inpainting: Transport

- Bertalmio, Sapiro, Caselles and Ballester (2000) were the first to apply *high-order* PDEs to inpainting: smoothness transportation

$$\boxed{\frac{\partial}{\partial t} u = \nabla^{\perp} u \cdot \nabla (L_{\text{smooth}}), \quad L_{\text{smooth}} \text{ can be } \Delta u.}$$

If the solution does converge as t → infinity, then L must remain constant along isophotes.



Andrea Bertozzi et. al (2001) found the connection to the Navier-Stokes and vortex dynamics for incompressible flows: treating u as the stream function.

We take a different approach ➡

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Our Approach: Bayes/Helmholtz Principle

- Inpainting is an image restoration problem.

- The universal approach for image restoration (denoising, deblurring, segmenting, e.t.c.) is the *Bayesian framework*. Or, in terms of machine and human vision, the *Helmholtz principle*.

- Bayesian MAP (maximum *a posteriori* probability) is to maximize

$$\boxed{\text{Prob}(u \mid u^0) = \text{Prob}(u^0 \mid u) \quad \times \quad \text{Prob}(u) \quad \text{(up to a constant)}}$$

The best guess u

is based on both

the way the observation u^0 is connected to u and,

the *a priori* popularity of the guess itself.

MAP: Maximum A Posteriori Probability

Bayesian framework for image restoration:

- ***Prior model***: **Prob(u) – What are images really?**
- ***Data model***: **Prob(u^0 | u) -- How is the observation u^0 generated from the ideal image u.**
- ***Bayes' Formula***:

$$\boxed{p(u \mid u^0) = \frac{p(u)}{p(u^0)} p(u^0 \mid u).}$$

- ***Best guess*** **= Maximum A Posteriori Probability:**

$$\boxed{\max p(u \mid u^0).}$$

Bayesian Goes Variational

Mumford (1994), "The Bayesian rationale for energy functionals,"

Bayesian formulation: $\max p(u \mid u^0).$

$$\boxed{p(u \mid u^0) = \frac{p(u)}{p(u^0)} p(u^0 \mid u).}$$

Energy (or variational) formulation:

$$\min E[u \mid u^0]$$

$$\boxed{E[u \mid u^0] = E[u] + E[u^0 \mid u].}$$

They are formally bridged by Gibbs' Law in Stat. Mechanics:

$$\boxed{\text{Probability} \propto \exp(-\text{Energy} / \kappa T).}$$

In this talk, we always use the energy/variational formulation.

Data Model Is Simple. Prior Model Crucial

- For most inpainting problems, the **data model** is simple :

$$u^0\,|_{\Omega \setminus D} = [K * u_{\text{original}} \oplus \text{noise}]_{\Omega \setminus D}\,.$$

Assuming Gaussian noise, then

$$E[u^0 \mid u] = \frac{\lambda}{2}\int_{\Omega \setminus D} (K * u - u^0)^2\,dx\,.$$

- Therefore, an effective Bayesian/variational inpainting model crucially depends on a good **(prior) image model** E[u] !

Geometric Image *Prior* Models

Ways to acquire *prior* image models:

- Markov/Gibbs random fields **(Geman-Geman, 1984; Blake-Zisserman, 1987; Black-Rangarajan, 1994)** based on the **lattice model** in Statistical Mechanics.
- Filtering and entropy based learning **(Zhu-Wu-Mumford, 1997, 1998).**
- Axiomatic approach for stochastic models **(Mumford-Gidas, 2000).**

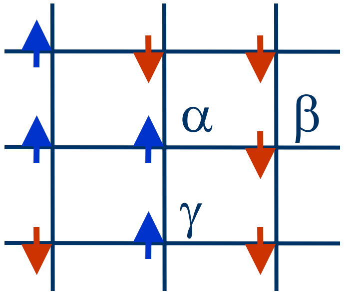
- **Geometric** models (in this talk):
  - A) **Bounded variation** (Rudin-Osher-Fatemi,1992,1994; Chan-Shen, 2000);
  - B) **The object-boundary model** (Mumford-Shah, 1989);
  - C) **Functionalized elastica** (Masnou-Morel, 1998, Chan-Kang-Shen, 2001);
  - D) **Mumford-Shah-Euler image** (Esedoglu-Shen, 2001).

First get a taste from the Ising Spin Model

Ising's Spin Crystal

- Ising's Lattice Spin Model (simplified ferromagnet):

  Spin up: $s = 1$; down: $s = -1$.

  $$E[s] = -\sum_{\alpha \propto \beta} J_{\alpha\beta} s_\alpha s_\beta - H \sum_\alpha s_\alpha .$$

  short range coupling — external field

  Ground state: s==sign(H).

  - 1-D model was solved by Ising (1925).
  - 2-D model by Onsager (1944).
  - Analytic solutions to (>2)-D models are still unknown.

  - First connected to vision/image analysis by Geman-Geman (Division Appl. Math., Brown U., 1984).

Inpainting **Binary** Images by Ising's Model

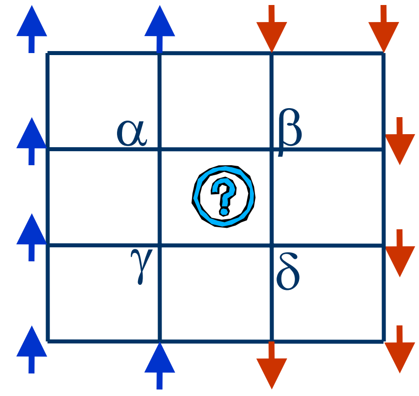
Suppose: boundary spins are known (locked). What are the spins at $\alpha, \beta, \gamma, \delta$? Assuming that there is no external field (i.e., H=0).

$$\min E[s_\alpha, s_\beta, s_\gamma, s_\delta \mid \text{given boundary spins}].$$

Solution for this example: $s_\alpha = s_\gamma = 1$; $s_\beta = s_\delta = -1$.

A **step-edge** is perfectly recovered ! ***However***,

- **Real images are generally not binary.**
- **Available image data are often polluted (by noise or blur).**
- **Geometry is not explicitly imposed. As a result, the regularity of the transition edges is generally not guaranteed.**



An Example

Geometry ? But How ?

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Functions with Bounded Variations (BV)

- BV(Ω)={ u | integrable and with finite total variation TV[u] }:

$$\mathrm{TV}\,[u] = \int_\Omega |Du| = \sup_{\text{smooth } \mathbf{f}:|\mathbf{f}|\le 1} \int_\Omega u \nabla \cdot \mathbf{f}\, dx\,.$$

The Sobolev space W(1,1) is its subspace, for which

$$\mathrm{TV}\,[u] = \int_\Omega |\nabla u|\, dx = \int_\Omega \sqrt{u_{x_1}{}^2 + u_{x_2}{}^2}\, dx\,.$$

Generally, TV is a Radon measure.

- **Geometry** of TV (why good for vision/image modeling):

Coarea Formula **(De Giorgi, 1961)**

$$E[u] = \int_{-\infty}^{\infty} \mathrm{Per}(u < \lambda, \Omega)\, d\lambda \overset{\text{smooth } u}{\Rightarrow} \int_{-\infty}^{\infty} \mathrm{length}(u = \lambda)\, d\lambda.$$

A collective way to impose geometry on all level-sets/edges !

TV Inpainting: Model & Computation

Chan-Shen (2000; *SIAM J. Appl. Math.*, 62(3), 2001)

The TV inpainting model: total variation (TV) energy

$$\boxed{\min_u \; E[u \,|\, u^0, D] = \int_\Omega |Du| + \frac{\lambda}{2}\int_{\Omega\backslash D} |u - u^0|^2 \, dx,}$$

least square (for Gaussian)

The associated *formal* Euler-Lagrange equation on Ω:

$$0 = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right] + \lambda_D(x)(u^0 - u), \quad \lambda_D(x) = \lambda \cdot 1_{\Omega\backslash D}(x).$$

with Neumann adiabatic condition along the boundary of Ω.

TV Inpainting: Existence

Chan-Kang-Shen [*SIAP*, 2002]

Existence Theorem for TV Inpainting:

$$\boxed{\min_u \; E[u \mid u^0, D] = \int_\Omega |Du| + \frac{\lambda}{2}\int_{\Omega\setminus D} |u - u^0|^2 \, dx,}$$

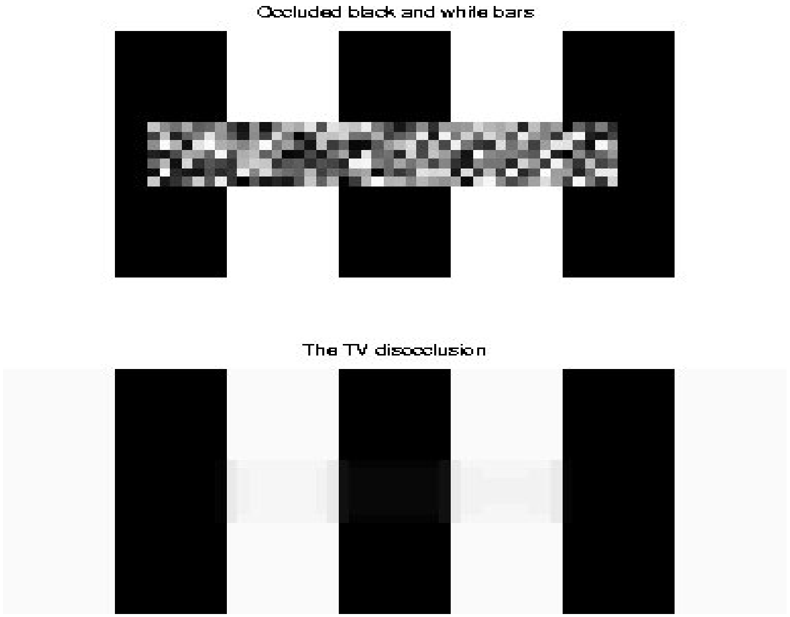
There **exists** at least one optimal inpainting in the space BV(Ω).

Proof. Similar to Chambolle and Lions (1997). *Applying*

- ***Lower semicontinuity & weak compactness.***
- ***Lebesgue dominated convergence theorem.***

TV Inpainting: An Example for Disocclusion

Chan-Shen [*SIAP*, 2001]



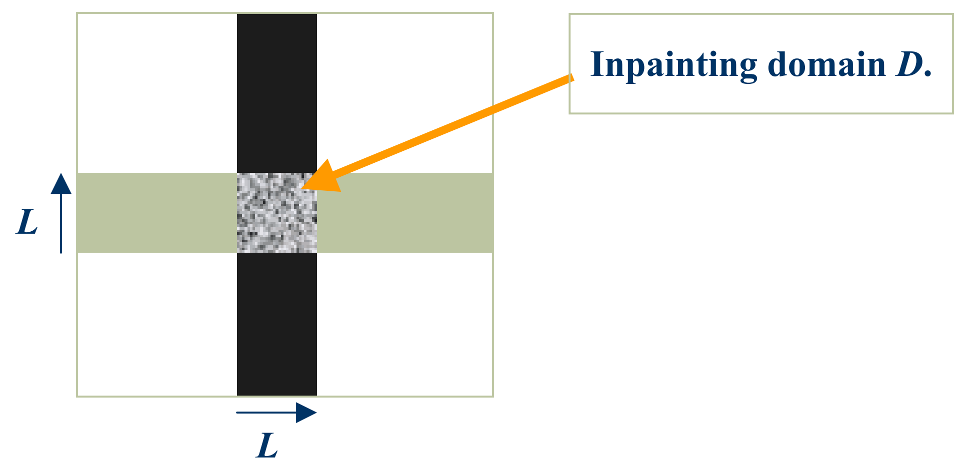
TV Inpainting: Uniqueness Is NOT Guaranteed

Chan-Kang-Shen [*SIAP*, 2002]

- Unlike Rudin-Osher-Fatemi's denoising model, the uniqueness of TV inpainting is generally **not** guaranteed.

- Non-uniqueness of the model, in our opinion, should be appreciated, instead of being cursed. It models the **multiple** valleys of the Bayesian decision/cost function, which simulates the uncertainty of human decision making.

- An example of uncertainty (vision foundation for non-uniqueness):



Viscosity Approximation of TV Inpainting

Computationally, the degenerated 2nd order Euler-Lagrange eqn. is solved by ***viscosity*** approximation (Osher-Sethian, Evans-Spruck),

$$0 = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|_\varepsilon} \right] + \lambda_D(x)(u^0 - u), \quad \lambda_D(x) = \lambda \cdot 1_{\Omega \setminus D}(x).$$

$$|a|_\varepsilon = \sqrt{a^2 + \varepsilon^2}$$

In terms of the variational formulation, this is to minimize

$$E_\varepsilon[u \mid u^0, D] = \int_\Omega \sqrt{|Du|^2 + \varepsilon^2} + \frac{\lambda}{2} \int_{\Omega \setminus D} |u - u^0|^2 \, dx.$$

Define $v = \varepsilon z - u$ (same for v_0), $x_\varepsilon = (x, z)$, $\Omega_\varepsilon = \Omega_\varepsilon$ x(0,1) (s. f. D_ε). Then,

$$E_\varepsilon[u \mid u^0, D] = E[v \mid v^0, D_\varepsilon] = \int_{\Omega_\varepsilon} |Dv| + \frac{\lambda}{2} \int_{\Omega_\varepsilon \setminus D_\varepsilon} |v - v^0|^2 \, dx_\varepsilon.$$

(A *thin-film* approximation)

Inpainting of Clean Images & Minimal Surface Problem

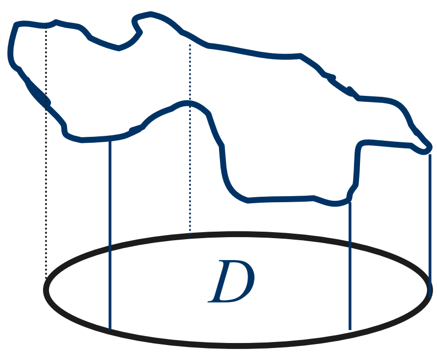
- Inpainting of clean (i.e. noise free) images (viscosity version):

$$\min_{u \in BV(D)} \int_D \sqrt{|Du|^2 + \varepsilon^2}, \quad \text{subject to} \quad u|_{\partial D} = u^0.$$

Fidelity of boundary data is infinity: λ=inf

inpainting domain only

in the sense of trace



- The classical (non-parametric) **minimal surface problem** (Giusti):

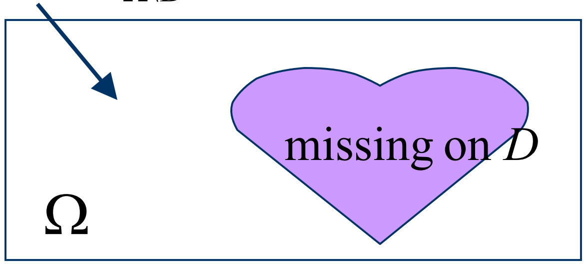
$$\min_{v \in BV(D)} A(v;D) = \int_D \sqrt{|Dv|^2 + 1}, \quad \text{subject to} \quad v|_{\partial D} = \varphi.$$

Minimize the total surface area of the graph

TV Inpainting of Blurred Images

$$u^0|_{\Omega\setminus D} = (K * u \oplus n)|_{\Omega\setminus D}$$

Chan-Shen (*AMS Contemp. Math.*, 2002)



The TV inpainting model:

Linear lowpass filter (blur)

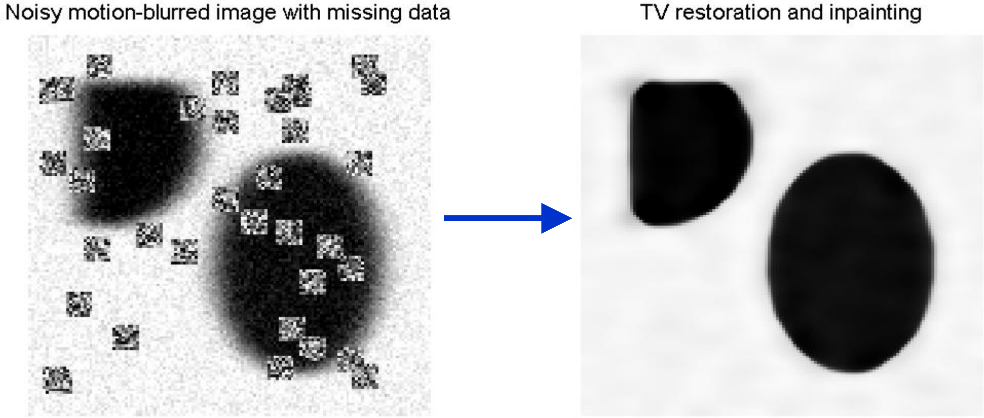
$$\min_u \quad E[u \mid u^0, D, K] = \int_\Omega |Du| + \frac{\lambda}{2}\int_{\Omega\setminus D} |K * u - u^0|^2 \, dx,$$

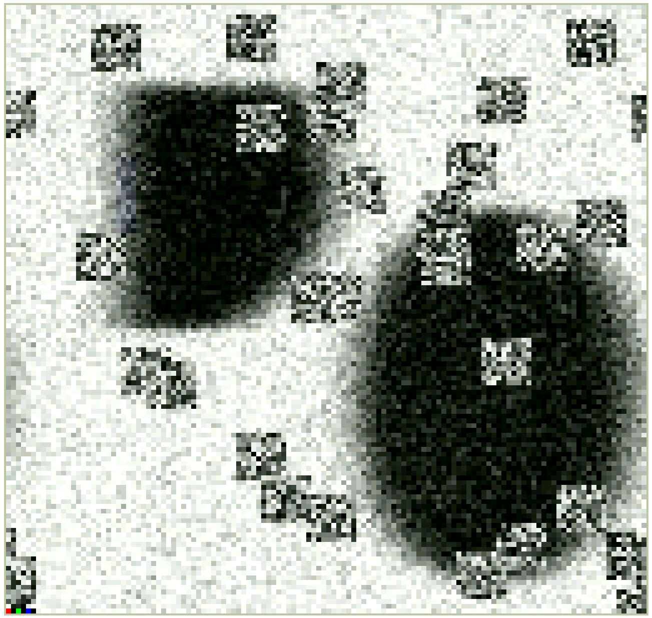
The associated *formal* Euler-Lagrange equation on Ω:

$$0 = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right] + K^t * \lambda_D(x)(K * u^0 - u), \quad \lambda_D(x) = \lambda \cdot 1_{\Omega\setminus D}(x).$$

with Neumann adiabatic condition along the boundary of Ω.

TV Inpainting for Noisy and Blurry Images





Chan-Shen
(*AMS Contemporary Math.*, 2002)

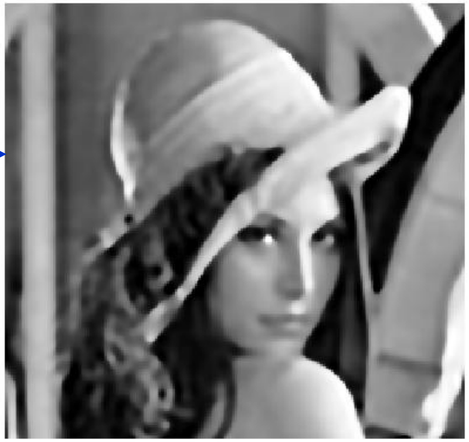
movie forever

Suppose $K=G_t$, is the **Gaussian** kernel. Then, the model gives a good inverting of heat diffusion. Without the TV regularization, **backward diffusion** is notoriously ill-posed.

TV Inpainting for the Error Concealment in Wireless Communication

A blurred image with 80 lost packets

Deblurring and error concealment by TV inpainting

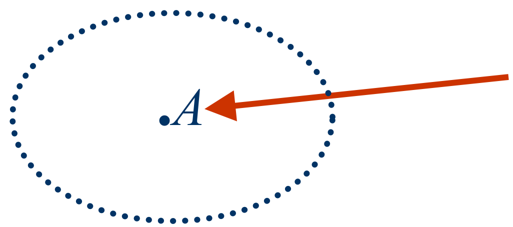


A Blurry Image With Lost Packets

Chan-Shen
(*AMS Contemporary Math.*, 2002)

movie once

Digital or Analog (i.e. Discrete vs. Continuous) ?



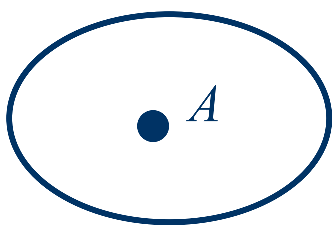
$u(A)=a$ **given on an interior pixel.**

Suppose that a is highly credible, i.e., no noise and no blurring.

A good inpainting scheme must take advantage of this exta information.

$u=\phi$ **given on surrounding pixels**

Theoretical crisis in the continuous (analog) interpolation theory:



$\Delta u=0, \quad u=\phi,$ along $\Gamma; \quad u(A)=a,$

which is ill-posed.

Remedy ? **Fattening** the inner pixel to an island. Clumsy ?

Another approach: Go completely digital

Self-Contained Graph Spectral Theory

Chung-Yau (1994,1995)

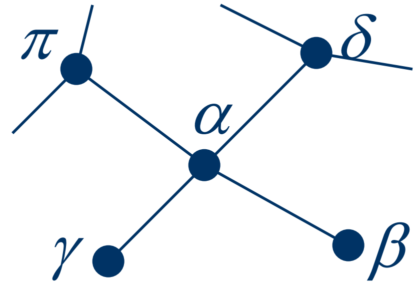
- Continuous case*:*

$$E[u] = \frac{1}{2}\int_D |\nabla u|^2 dx \xrightarrow{\text{gradient}} \Delta u = -\frac{\partial E}{\partial u}.$$

- Graph Laplacian *(d is the degree of a node):*

$$E_g[u] = \tfrac{1}{2}\sum_{\alpha\propto\beta}(u_\alpha - u_\beta)^2 \xrightarrow{\text{gradient}} \Delta_g u\Big|_\alpha = -du_\alpha + \sum_{\beta\propto\alpha} u_\beta,$$

which *encodes* all the information of the underlying graph.



Self-Contained Digital (Graph) TV Theory

Chan-Osher-Shen (2001)

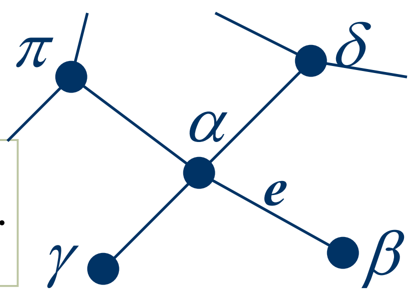
- Continuous case*:*

$$\mathrm{TV}[u] = \int_D |\nabla u|\,dx \xrightarrow{\text{gradient}} \kappa = \nabla \bullet \left[\frac{\nabla u}{|\nabla u|}\right] = -\frac{\partial \mathrm{TV}[u]}{\partial u}, \text{ the curv.}$$

- Graph TV and graph curvature*:*

$$\mathrm{TV}_{\mathrm{g}}[u] = \sum_{\alpha \in G} |\nabla_\alpha u| \xrightarrow{\text{gradient}} k_\alpha = -\frac{\partial \mathrm{TV}_{\mathrm{g}}[u]}{\partial u_\alpha}.$$

$$|\nabla_\alpha u| = \sqrt{\sum_{\beta \propto \alpha} (u_\beta - u_\alpha)^2}.$$

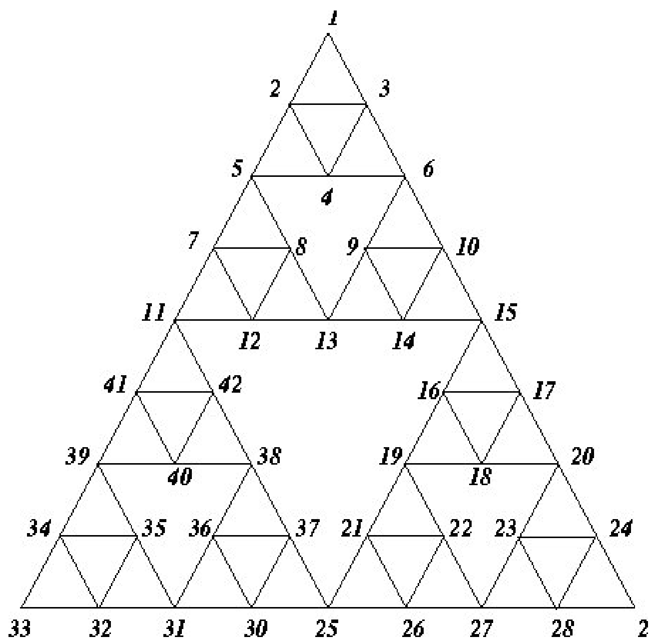


Beautiful formula:

$$k_\alpha = -\frac{\partial \mathrm{TV}_{\mathrm{g}}[u]}{\partial u_\alpha} = \sum_{e \propto \alpha} \frac{\partial}{\partial e} \frac{1}{|\nabla u|} \frac{\partial}{\partial e}\bigg|_\alpha .$$

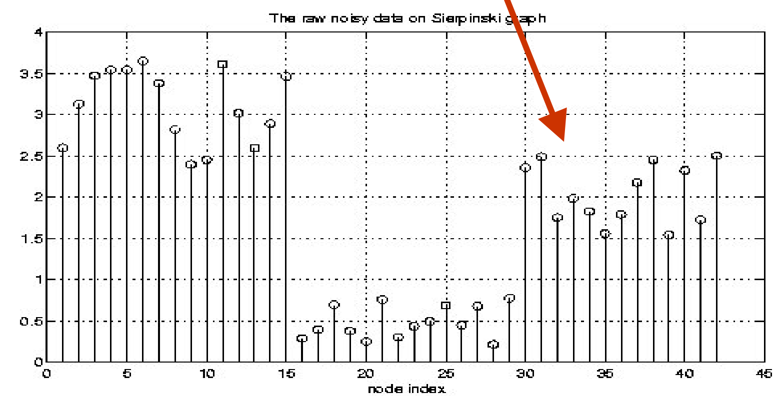
- For *weighted* graphs, weights can be incorporated.

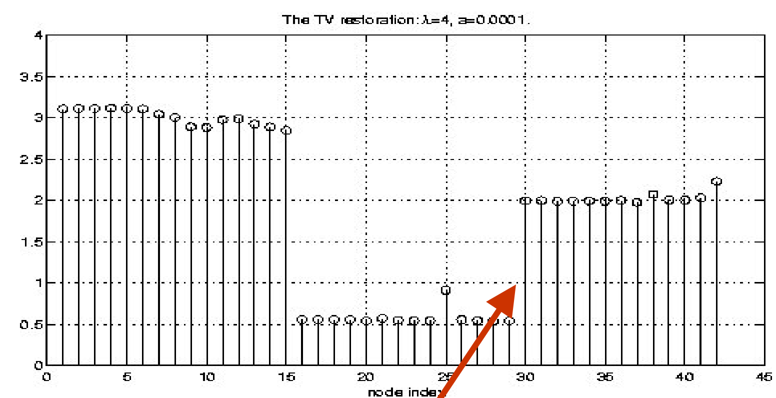
Digial TV Denoising of Data on Sierpinski Graph



The Sierpinski graph at level 3

Noisy data



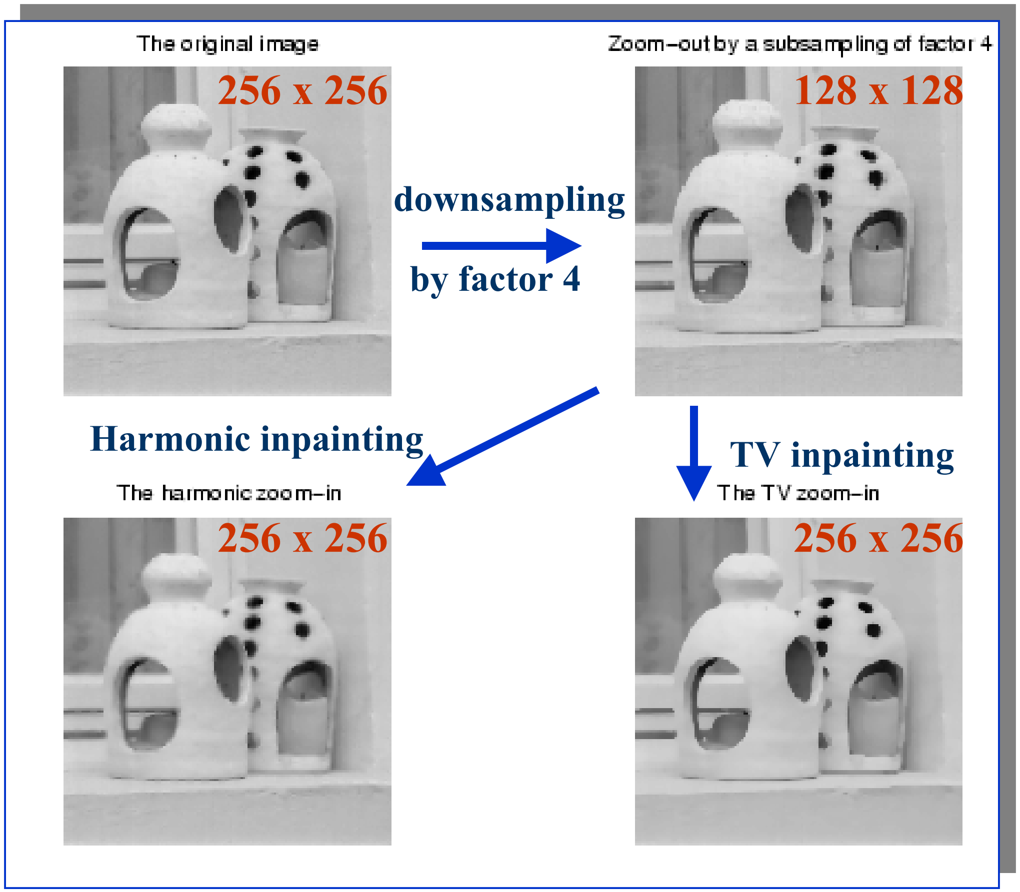


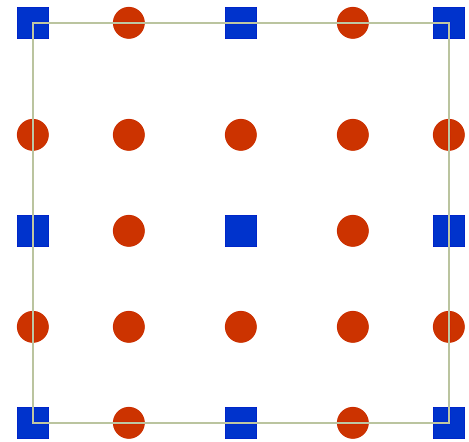
Sharp transition is not smeared

Digital Zoom-in by (Digital) TV Inpainting

(test image from Caltech Comp. Vision Lab)

Chan and Shen (*SIAP*, 2001)





■ Coarse scale

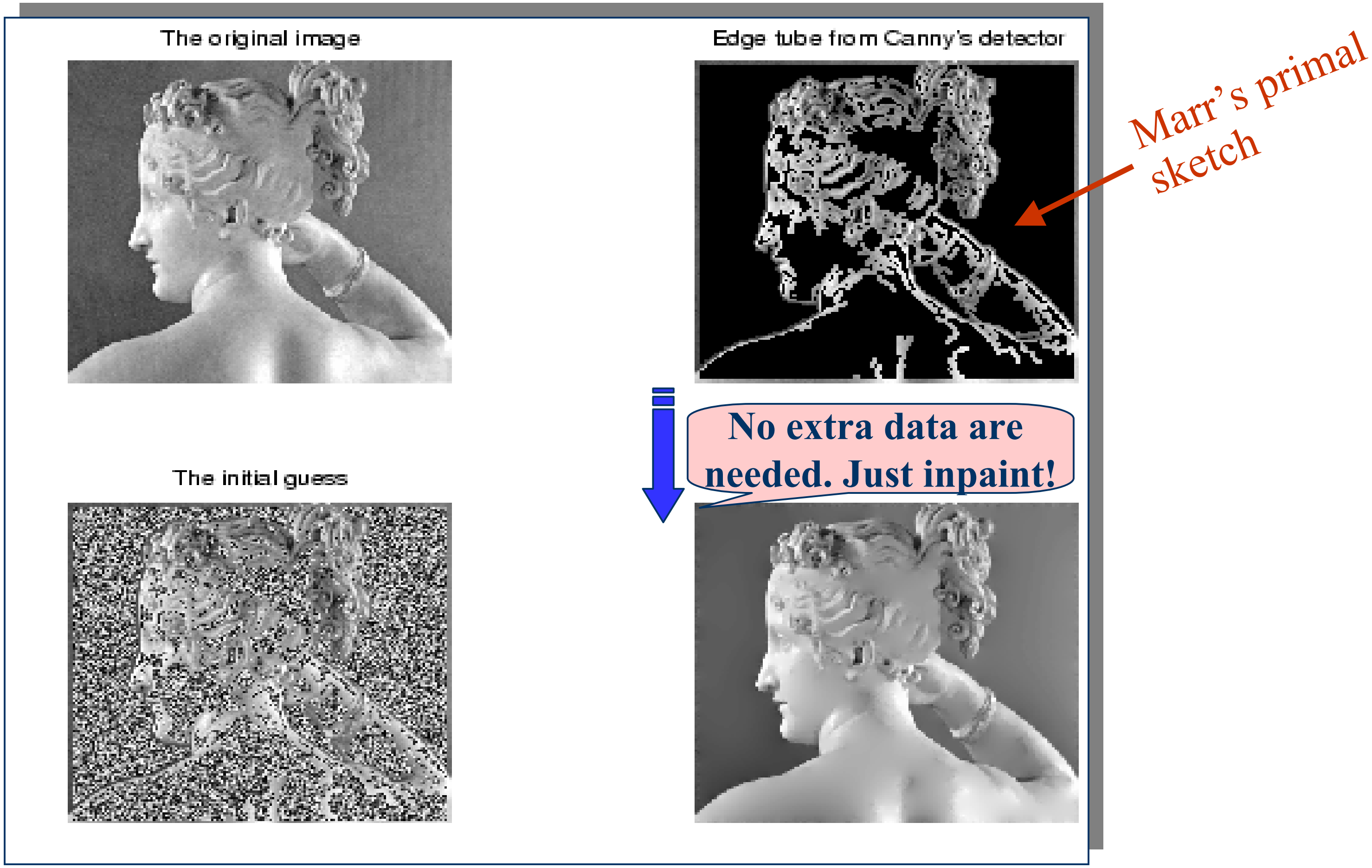
● Finer scale

Sharp edges are successfully inpainted by TV, but blurred by Sobolev norms.

Decoding Marr's Primal-Sketch by Digital TV Inpainting

(test image from Caltech Comp. Vision Lab)

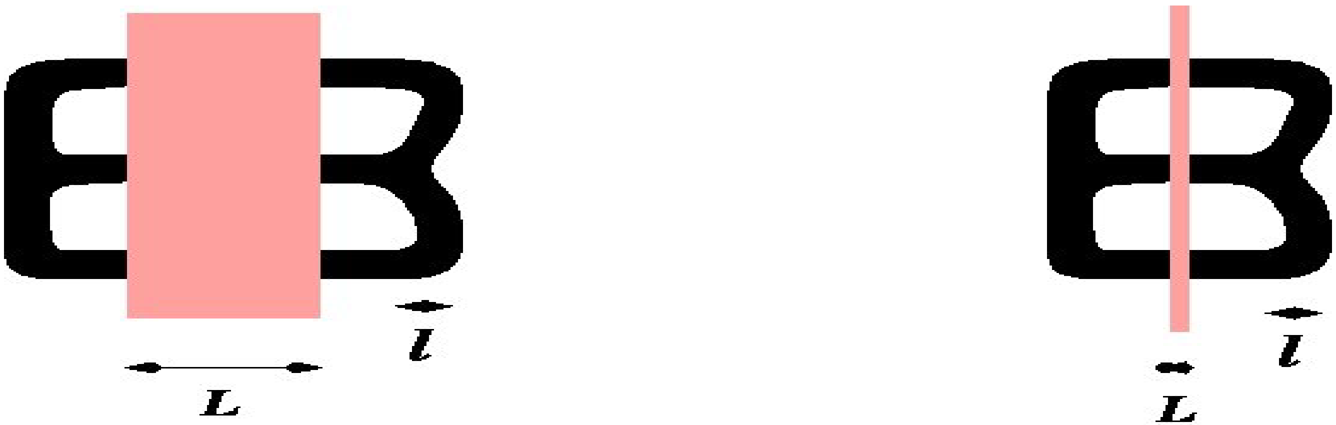
Chan and Shen (*SIAP*, 2001)



TV helps regularize the messy edge set

TV Inpainting & Human Visual Perception. I.

(Kanisza)

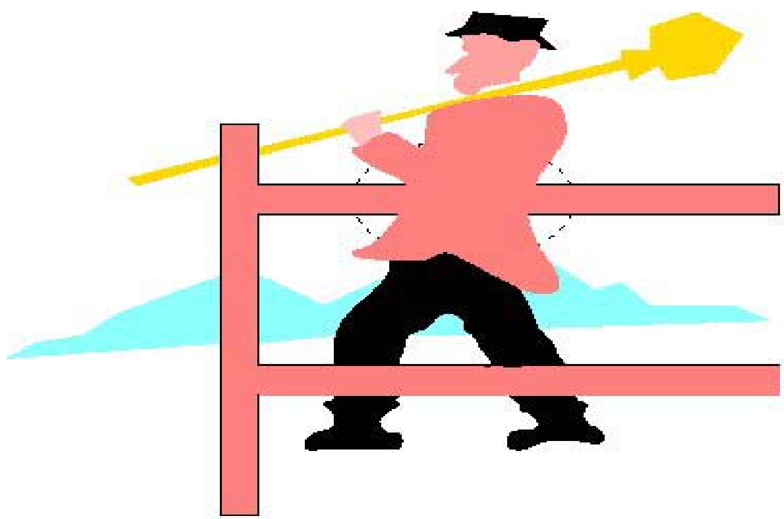


"E 3" or "B" ?

What we perceive (or guess) depends on the aspect ratio.
So is TV inpainting!

TV Inpainting & Human Visual Perception. II.

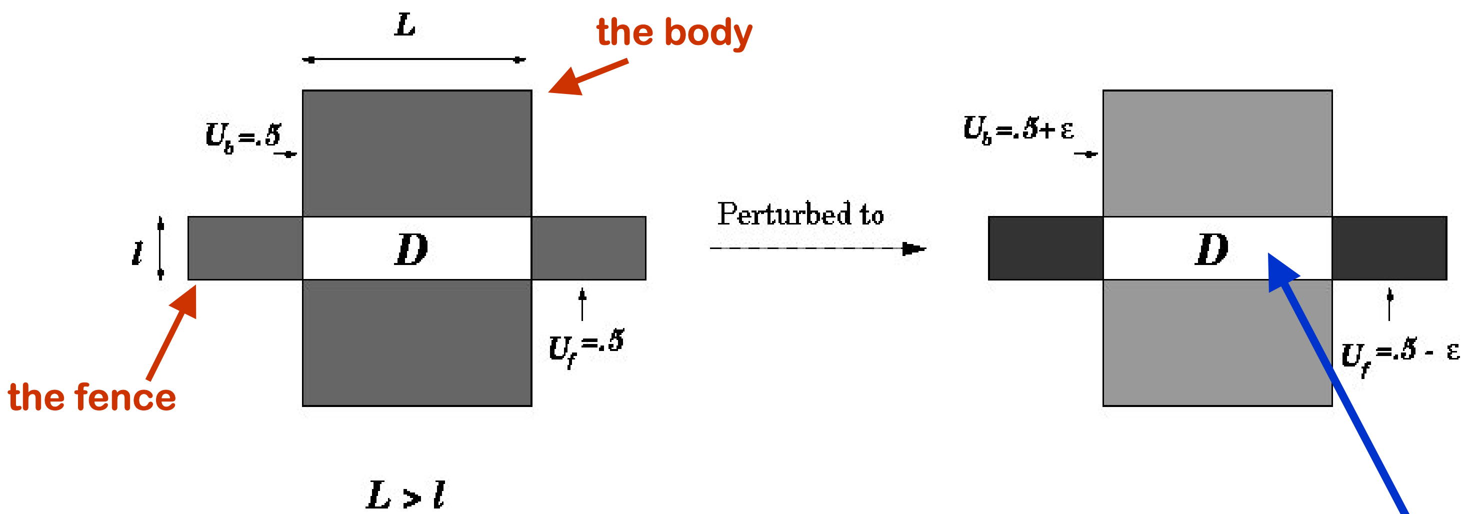
(Kanisza)



Kanizsa's entangled man

Can TV Inpainting Explain the Entangled Man?

Answer: Yes! It can.



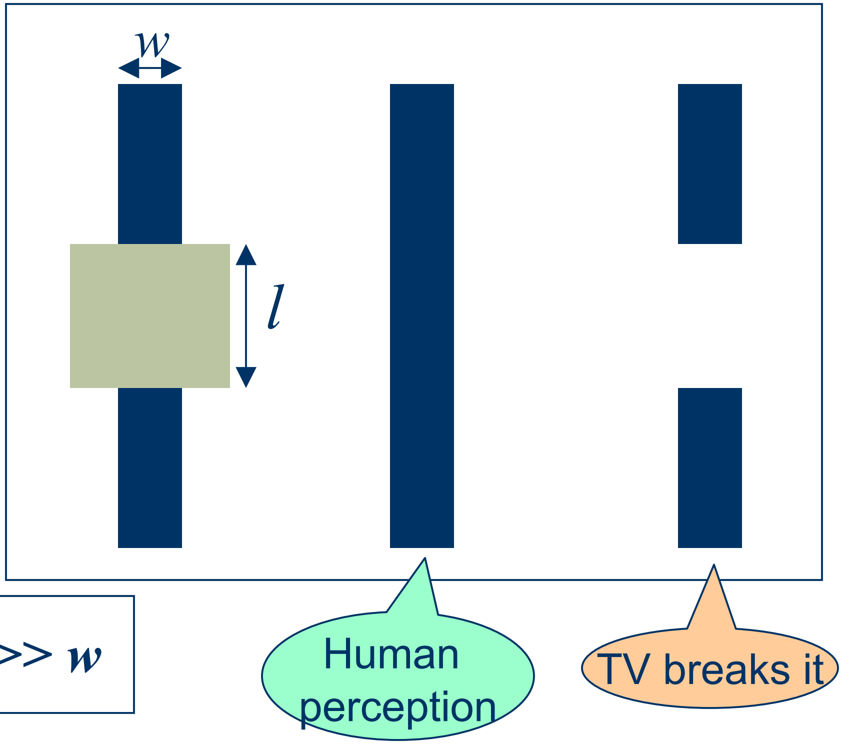
$$\mathrm{TV}(u_D) = [(1+2\varepsilon)L - (1-2\varepsilon)l] - (L-l)c$$

$$u_f = .5 - \varepsilon \le c \le .5 + \varepsilon = u_b$$

➡ **To minimize the TV norm,** c **= the body color =** $.5 + \varepsilon$**!**

TV & Human Visual Perception. III. TV is Insufficient

(Kanisza, Nitzberg-Mumford, Chan-Shen)



When $l >> w$

To fix the problem,
Chan and Shen (2001)
proposed the CDD inpainting.

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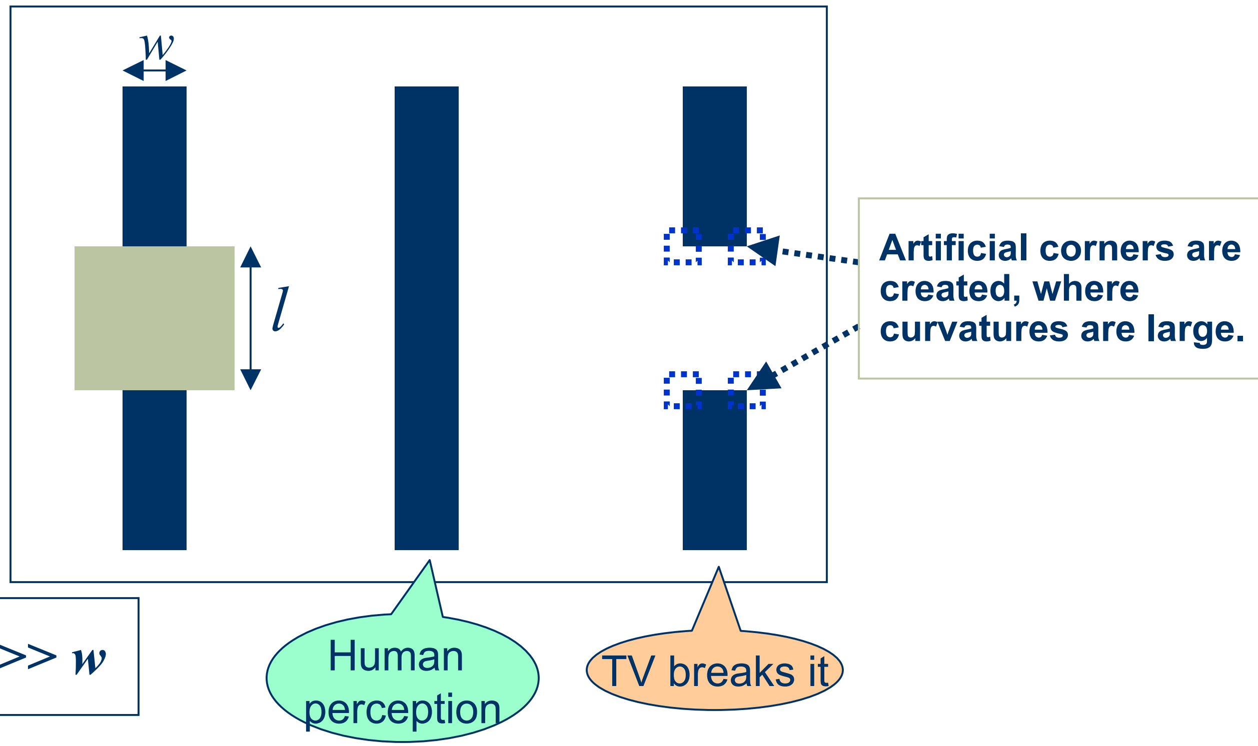
Connectivity Principle in Vision: TV is Insufficient

(Kanisza, Nitzberg-Mumford, Chan-Shen)



When $l >> w$

CDD Inpainting: Curcature Driven Diffusion

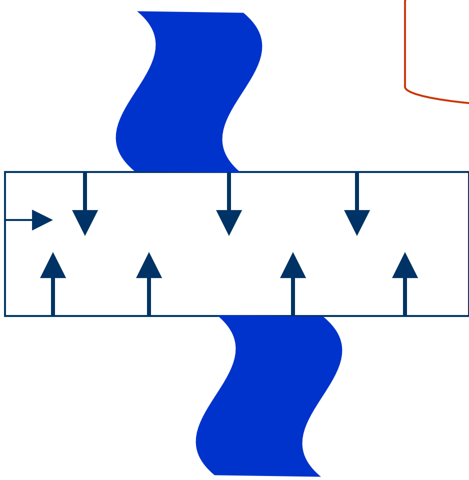
Chan and Shen (*J. Visual Comm. Image Rep.*, 2001)

- Chan and Shen introduced an inpainting mechanism based on CDD: *curvature driven diffusions*. Information is "fluxed into" into the inpainting domain by diffusions:

$$\frac{\partial}{\partial t} u = \nabla \cdot \left[\frac{F(x, |\kappa|)}{|\nabla u|} \nabla u \right] + \lambda_e(x)(u - u^0),$$

where, F in the diffusivity coeff is to penalize large curvature.

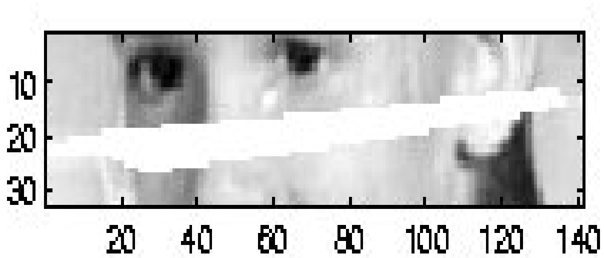
Large curvatures at artificial corners are penalized!

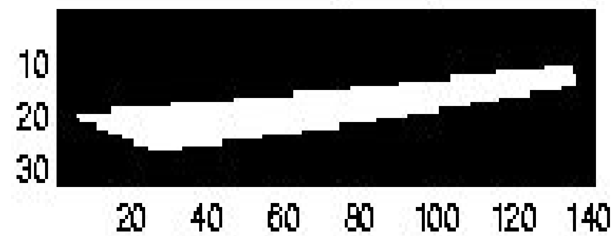


CDD generally encourages the connection of broken parts, and thus realizes the Connectivity Principle in vision.

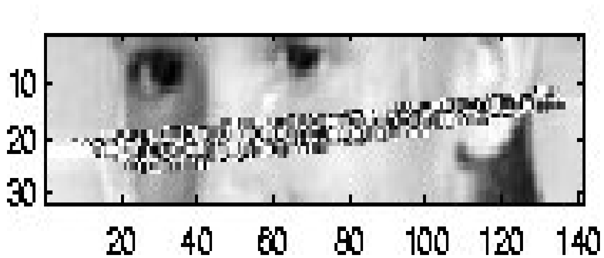
CDD Inpainting: Connection Enforcement

Chan and Shen (*J. Visual Comm. Image Rep.*, 2001)

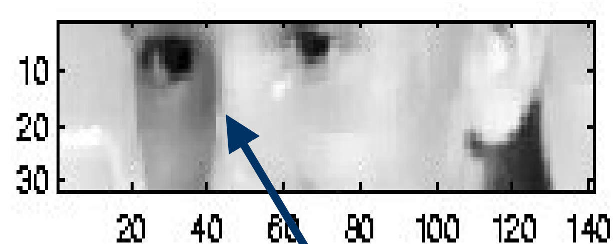




Inpainting mask



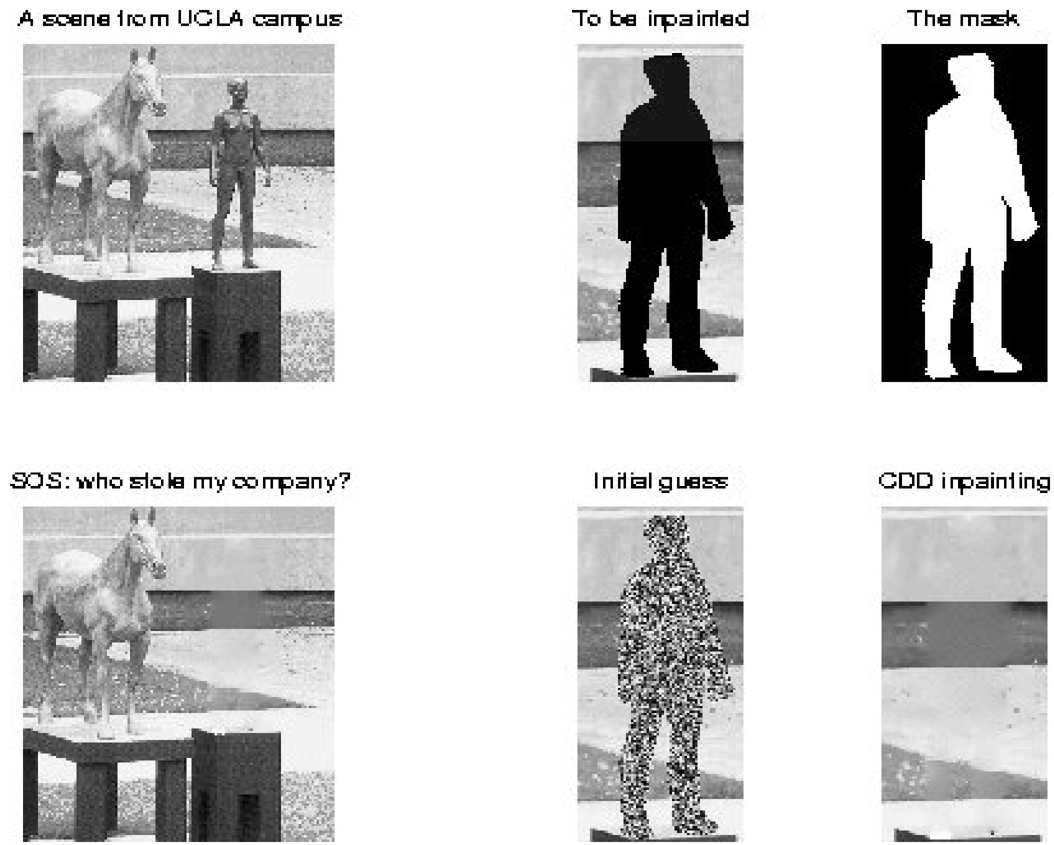
Random initial guess



Such weak edge is very nicely inpainted

CDD Inpainting: Who Stole My Company?

Chan and Shen (*J. Visual Comm. Image Rep.*, 2001)



inpainting
mask

From the courtyard of Rolfe Hall, UCLA campus

Combining CDD with Sapiro's Transport

Chan and Shen (*AMS Contemp. Math.*, 2002)

- Quasi-axiomatic approach to integrate the two microscopic inpainting mechanisms.

- Axioms (**Chan-Shen, 2002**):
  - **Morphological invariance**
  - **Rotational invariance**
  - **Forward (stable) diffusion**
  - **Linear interpolation for pure transport**

- Then, there is only one class of 3rd order PDEs for inpainting:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(f(\kappa, \sigma)\,\vec{n} + a\,\sigma\,\vec{t} \right).$$

$$f > 0, \quad \kappa = \nabla \cdot \vec{n}, \quad \sigma = \partial \ln|\nabla u| / \partial \vec{t}\,.$$

Our next model further explores the role of curvature

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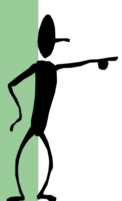
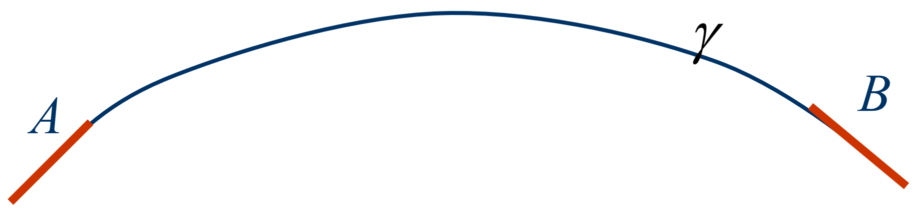


Once upon a time, there was a guy named

Euler

He studied how a thin and torsion free ***rod*** bends under external forces at the two ends (1744):



The energy that controls the shape is given by the total squared curvature :

$$e_2[\gamma] = \int_\gamma (a + b\kappa^2)\,ds\,.$$

The equilibrium curves (local minima) are called elasticas.

Elastica as *Nonlinear-splines*

- G. Birkhoff and C. De Boor (1965) suggested to apply elasticas as new interpolation tools, or *nonlinear splines,* contrast to linear cubic splines in approximation theory.

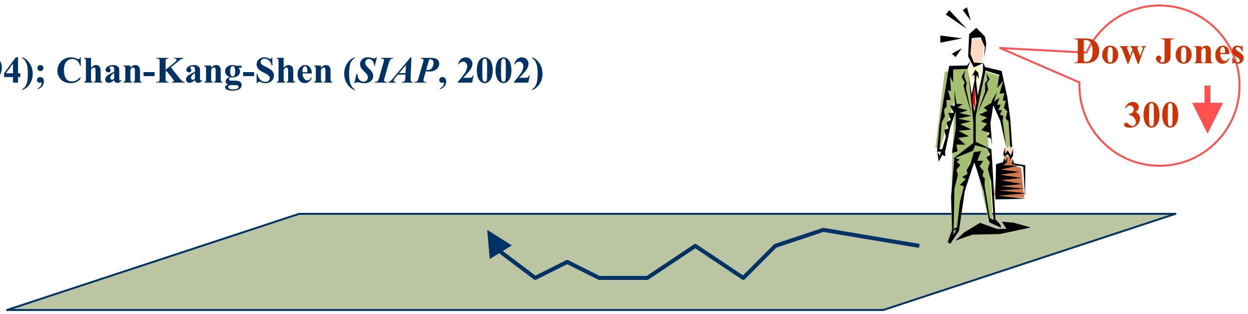
- D. Mumford (1994) first introduced Euler's elastica into computer vision, as a prior curve model, and expressed the solutions to the E-L equation in terms of elliptic functions:

$$\boxed{2\kappa''(s) + \kappa^3(s) = \frac{a}{b}\kappa(s).}$$

- Nitzberg, Mumford, and Shiota (1993) employed elasticas to connect large-scale occluded edges in vision modeling.

Elasticas and A Drunk's Walking Path: Statistical Meaning

Mumford (1994); Chan-Kang-Shen (*SIAP*, 2002)



The walking characteristics of the steps (fixed N steps):

- step sizes ($h_1, h_2, \ldots, h_N$) are i.i.d. of exponential type.
- the uncertainty of the turn angle made at each step k is <u>completely</u> determined by and <u>linearly proportional</u> to the step size h_k. The ratios are Gaussian i.i.d.'s with mean 0.

Then the distribution of the N-step polygonal walks γ :

$$\mathrm{pdf}\,(\gamma) = \frac{1}{Z}\exp\left(-\lambda L(\gamma) - \frac{1}{2\sigma^2}\left\|\kappa^2\right\|_\gamma\right) = \frac{1}{Z}\exp(-e_2(\gamma)).$$

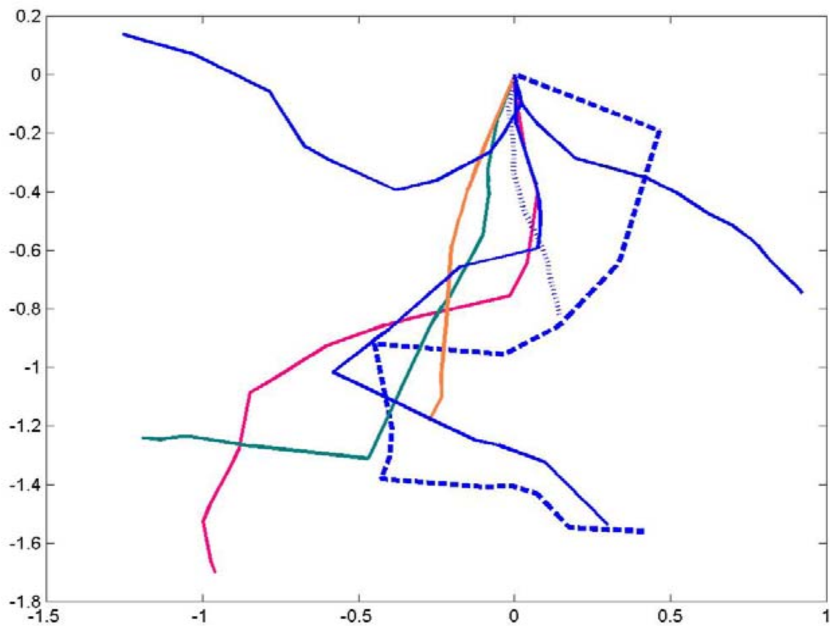
Sample Walking Paths of an Elastica Drunk

Chan-Kang-Shen (*SIAP*, 2002)

Samples of 20-step walks:

For step sizes: Exponential with mean 1/10;

For the turns : Gaussian with mean 0 and std=3.



MATLAB Simulation

Unlike *Brownian* motions, the paths of an elastica drunk are more regular (smoother), which is what computer vision prefers for most contours in our daily life: buildings, desks, computers, etc. (Fractal coast lines are exceptions.)

Lifting a Curve Model to an Image Model

- Using the level-sets of an image, we can "lift" a curve model to an image model (formally; theoretical study by Bellettini, et. al):

$$E_2[u] = \int_D (a + b\kappa^2)\,|\nabla u|\,dx$$
$$= \int_0^1 d\lambda \int_{\gamma_\lambda : u=\lambda} (a + b\kappa^2)\,ds$$
$$\kappa = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right] = \nabla \cdot \vec{n}$$

Notice that for the *mean curvature flow* (Evans, IPAM notes):

$$u_t = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\right) u_{x_i x_j},$$

$$\frac{d}{dt}\int_D |\nabla u|\,dx = -\int_D \kappa^2 |\nabla u|\,dx.$$

The Elastica Inpainting Model

Masnou-Morel (1998), Chan-Kang-Shen (*SIAP*, 2002)

$$E[u \mid u^0, D] = \int_\Omega \varphi(\kappa)\,|\nabla u|\,dx + \frac{\lambda}{2}\int_{\Omega \setminus D} |u - u^0|^2\,dx,$$

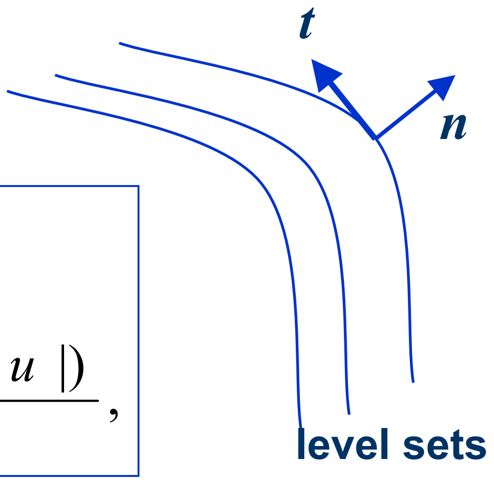
$$\kappa = \nabla \cdot \vec{n} \ \text{ is the curvature,} \quad \varphi(\kappa) = a + b\kappa^2.$$

Theorem (the associated PDE model).

The gradient descent flow is given by

$$\frac{\partial u}{\partial t} = \nabla \cdot \vec{V} + \lambda_D(x)(u^0 - u),$$

$$\vec{V} = \varphi(\kappa)\vec{n} - \frac{\vec{t}}{|\nabla u|}\frac{\partial(\varphi'(\kappa)|\nabla u|)}{\partial \vec{t}},$$



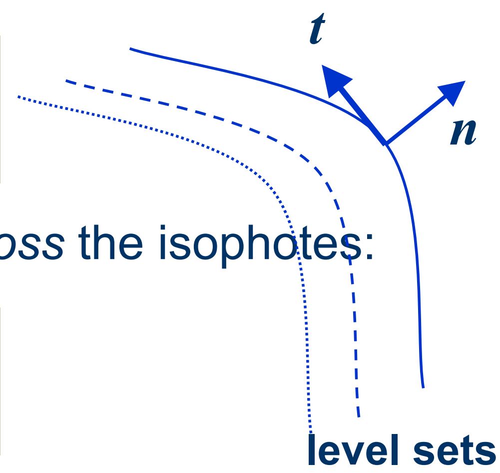
where V is called the flux field, with proper boundary conditions.

Elastica Inpainting: Also Tranport + CDD

Chan-Kang-Shen (*SIAP*, 2002)

– Transport *along* the isophotes:

$$\boxed{V_t = -\frac{\vec{t}}{|\nabla u|}\frac{\partial(\varphi'(\kappa)|\nabla u|)}{\partial \vec{t}},}$$



– Curvature driven diffusion (CDD) *across* the isophotes:

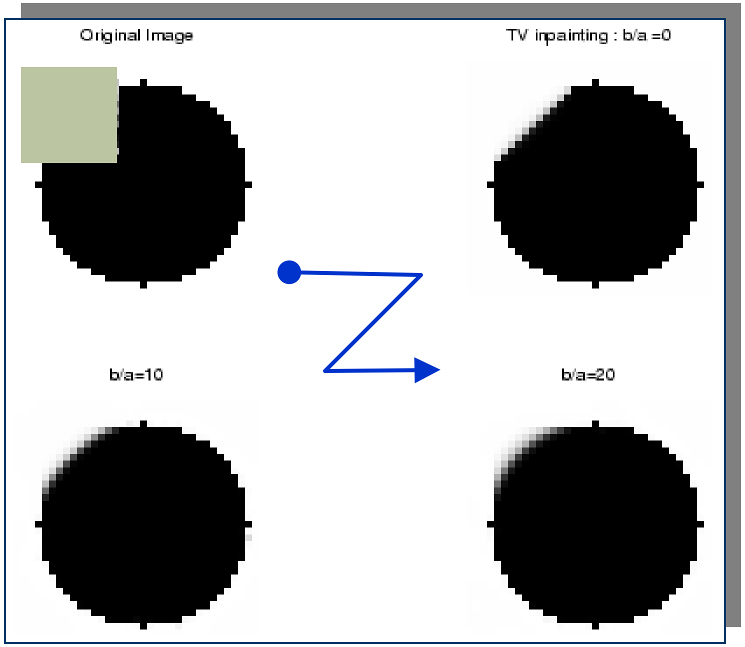
$$\boxed{V_n = \varphi(\kappa)\vec{n} = \varphi(\kappa)\frac{\nabla u}{|\nabla u|}.}$$

Conclusion:

Elastica inpainting unifies the earlier work of Bertalmio, Sapiro, Caselles, and Ballester (2000) on transport based inpainting, and that of Chan and Shen (2001) on CDD inpainting.

Elastica Inpainting. I. Smoother Completion

Effect 1: as b/a increases, connection becomes smoother.

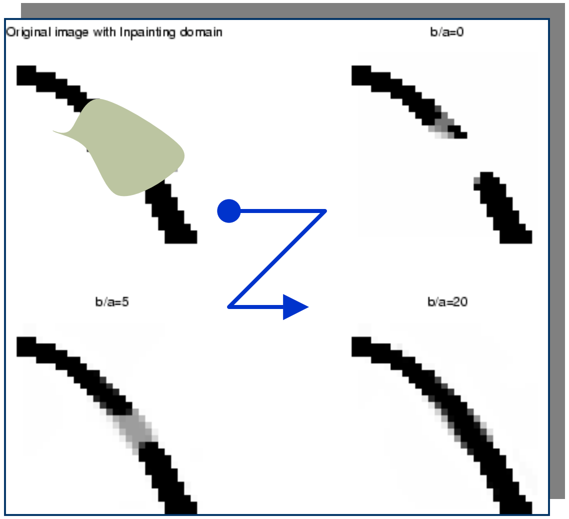


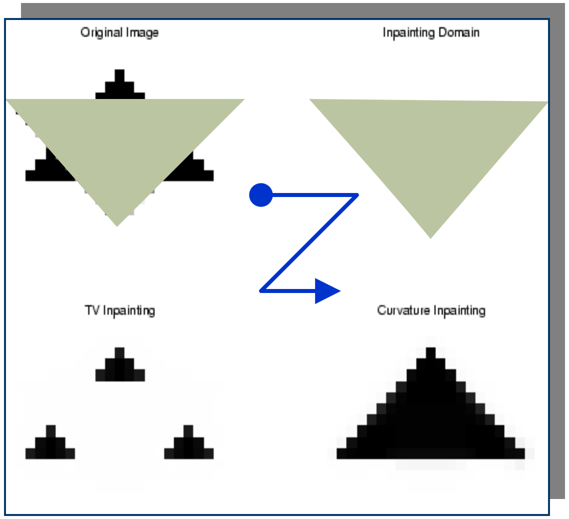
$$\text{Euler's elastica}: \phi(\kappa) = a + b\kappa^2.$$

Elastica Inpainting. II. Long Distance is Cheaper

Effect 2: as b/a increases,
long distance connection gets cheaper.

$$\text{Euler's elastica: } \phi(\kappa) = a + b\kappa^2.$$





For more theoretical and computational (4th order nonlinear!) details, please see Chan-Kang-Shen (*SIAP*, in press, 2002).

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Mumford-Shah Based Image Inpainting

Chan-Shen (*SIAP*, 2000), Tsai-Yezzi-Willsky (2001), Esedoglu-Shen (*EJAP*, 2002)

The Mumford-Shah (1989) image model was initially designed for the segmentation application:

$$E_{\mathrm{ms}}[u,\Gamma] = E[u \mid \Gamma] + E[\Gamma] = \frac{\gamma}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 \, dx + \alpha \ \mathrm{length}\ (\Gamma),$$

Mumford-Shah based inpainting is to minimize:

$$E_{\mathrm{ms}}[u,\Gamma \mid u^0, D, K] = E_{\mathrm{ms}}[u,\Gamma] + E[u^0 \mid u, \Gamma, D, \ K].$$

Inpainting domain

possible blurring

A free boundary optimization problem.

Mumford-Shah Inpainting: **Algorithm**

- For the **current** best guess of edge layout Γ, find u to minimize

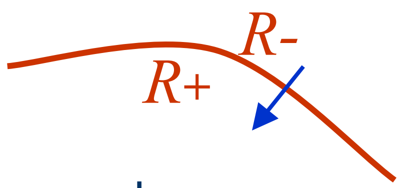
$$\boxed{E_{\mathrm{ms}}[u \mid \Gamma, u^0, D] = E[u \mid \Gamma] + E[u^0 \mid u, \Gamma, D] + const\,.}$$

→ equivalent to solving the elliptic equation on $\Omega \backslash \Gamma$:

$$\boxed{\gamma \Delta u + \lambda_e(x)(u^0 - u) = 0\,.}$$

- This updated guess of u then guides the motion of Γ:

$$\boxed{\frac{dx}{dt} = (\alpha\kappa + [R]_\Gamma)\vec{n}\,.}$$



M. C. Motion Jump across Γ of the roughness measure

$$\boxed{R = \tfrac{\gamma}{2}|\nabla u|^2 + \tfrac{\lambda_e}{2}(u - u^0)^2\,.}$$

[We can then benefit from the level-set implementation by Chan-Vese.]

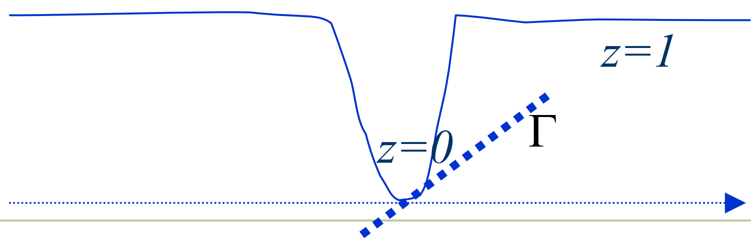
Mumford-Shah Inpainting via Γ-Convergence

Esedoglu-Shen (*Europ. J. Appl. Math.*, 2002)

The Γ-convergence approximation of Ambrosio-Tortorelli (1990):

$$E_{\mathrm{ms}}[u,\Gamma] = \frac{\gamma}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2\, dx + \alpha\ \mathrm{length}\,(\Gamma),$$

$$E_\varepsilon[u,z] = \frac{\gamma}{2}\int_\Omega (z^2 + o(\varepsilon))\,|\nabla u|^2 dx + \alpha\int_\Omega \left(\varepsilon\,|\nabla z|^2 + \frac{(1-z)^2}{4\varepsilon}\right) dx\,.$$



edge Γ is approximated by a signature function z.

Esedoglu-Shen shows that inpainting is the perfect market for Γ-convergence

Simple **Elliptic** Implementation

Esedoglu-Shen (2002)

The associated equilibrium PDEs are two coupled *elliptic* equations for u and z, with Neuman boundary conditions:

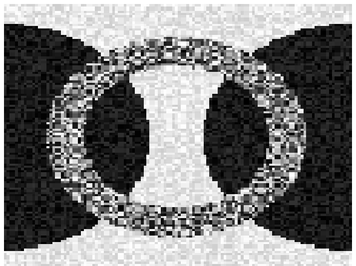
$$\boxed{\begin{aligned} &\lambda_D(x)(u-u^0)-\gamma\nabla\cdot(z^2\nabla u)=0, \\ &(\gamma\,|\nabla u|^2)z+\alpha\left(-2\varepsilon\Delta z+\frac{z-1}{2\varepsilon}\right)=0, \end{aligned}}$$

which can be solved numerically by any efficient elliptic solver.

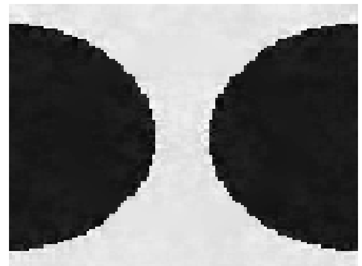
Applications: Disocclusion and Text Removal

Esedoglu-Shen (2002)

Noisy image to be inpainted

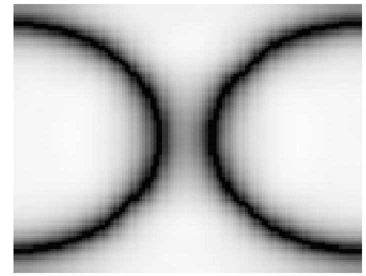


Inpainting output u



inpainted u

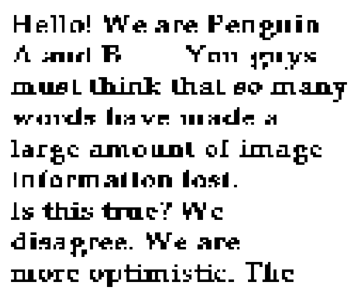
Inpainting output z



the edge signature z

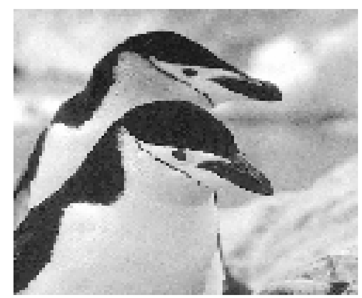
Image to be inpainted

Inpainting domain (or mask)



Inpainting domain

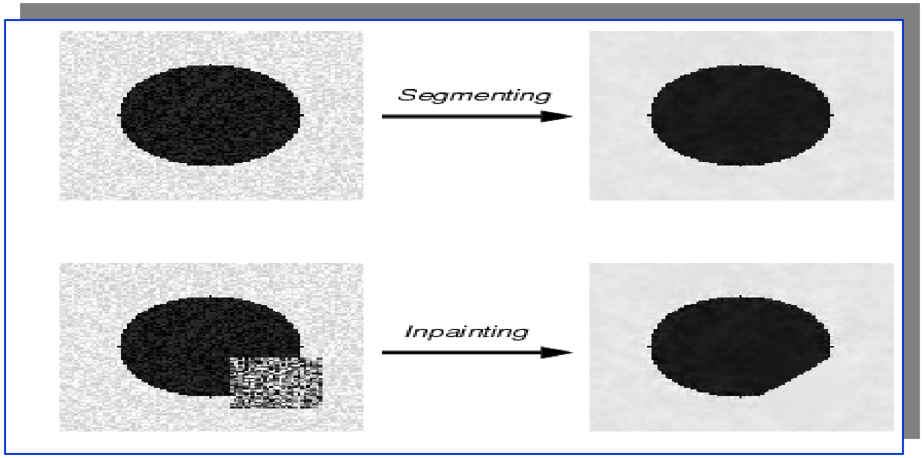
Inpainting output



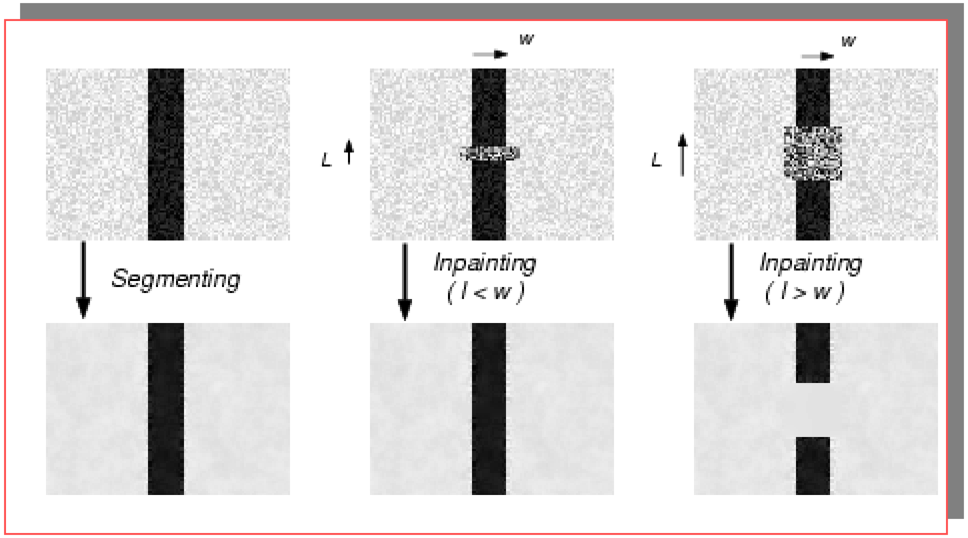
inpainted u

Insufficiency of Mumford-Shah Inpainting

Defect I:
Artificial corners



Defect II:
Fail to realize
the Connectivity
Principle, like TV.



Mumford-Shah-Euler Inpainting

Esedoglu-Shen (2002)

Idea: change the *straight-line* curve model embedded in the Mumford-Shah image model to *Euler's elastica:*

$$E_{\mathrm{mse}}[u,\Gamma] = \frac{\gamma}{2}\int_{\Omega\backslash\Gamma} |\nabla u|^2 \, dx + e(\Gamma),$$

$$e(\Gamma) = \alpha \ \mathrm{length}(\Gamma) + \beta \int_\Gamma \kappa^2 ds, \ \text{the elastica energy.}$$

The Γ-convergence approximation (**conjecture**) of De Giorgi (**1991**):

$$E_\varepsilon[u,z] = \frac{\gamma}{2}\int_\Omega (z^2 + o(\varepsilon))\,|\nabla u|^2 dx + \alpha \int_\Omega \left(\varepsilon\,|\nabla z|^2 + \frac{W(z)}{4\varepsilon} \right) dx$$

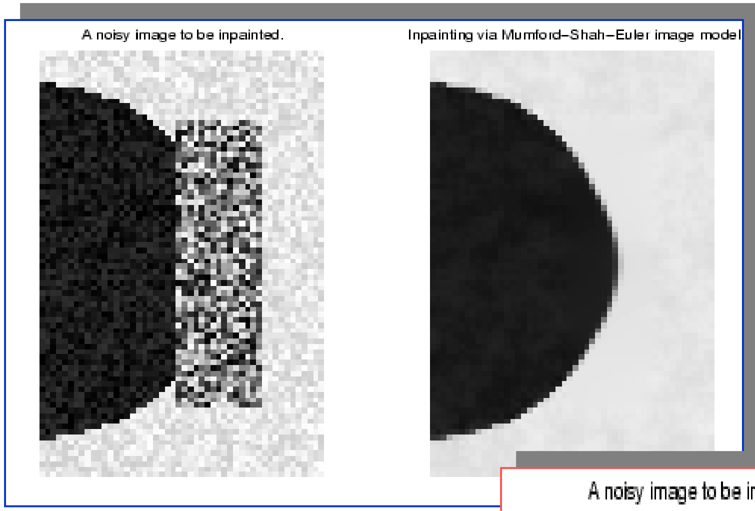
$$+ \frac{\beta}{\varepsilon}\int_\Omega \left(2\varepsilon \Delta z - \frac{W'(z)}{4\varepsilon} \right)^2 dx,$$

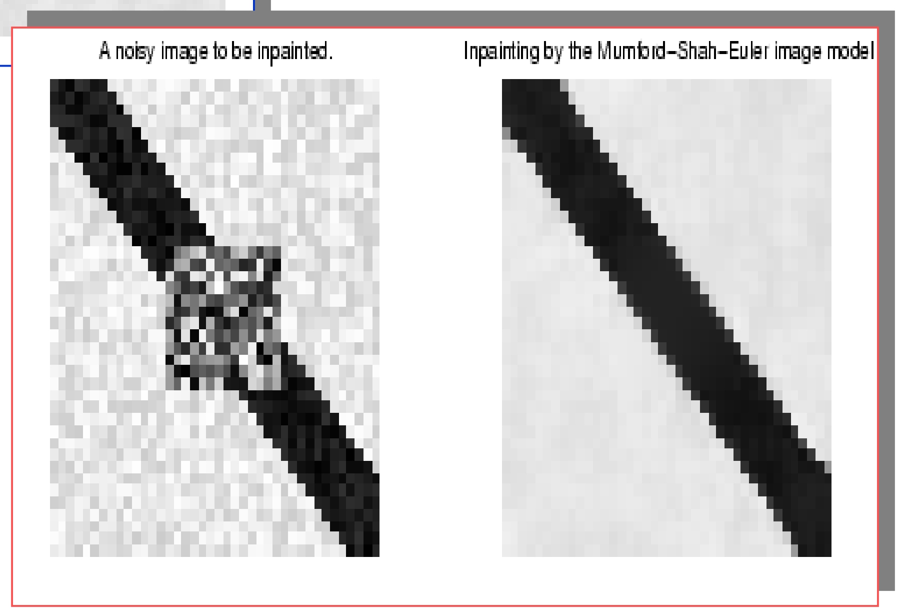
$$W(z) = (1-z)^2(1+z)^2 \ \text{is the double - well potential.}$$

For the technical and computational details, please see Esedoglu-Shen.

Features of Mumford-Shah-Euler Inpainting

Esedoglu-Shen (2002)





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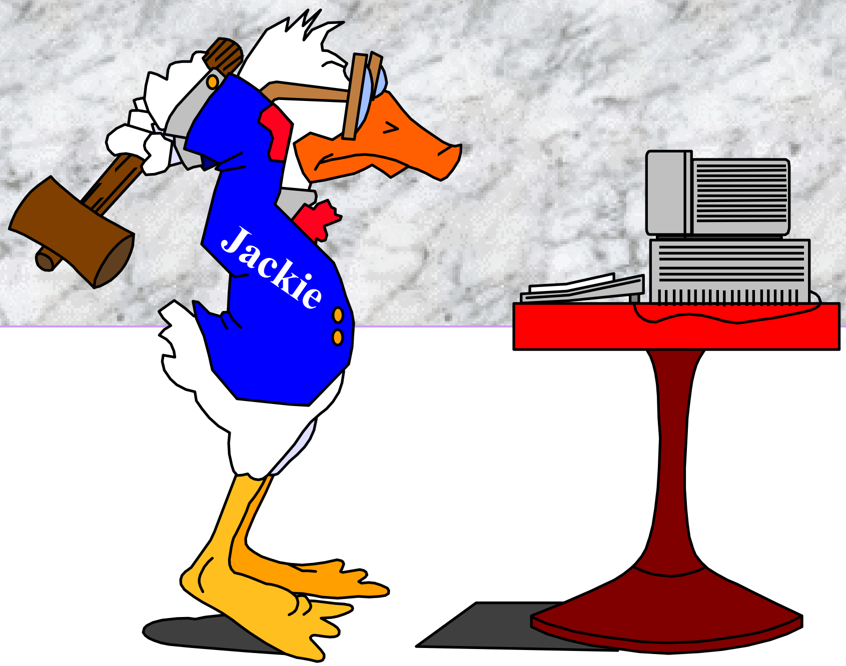
Conclusion

- Bayesian framework (or Helmholtz principle) is the foundation of our approach to inpainting/visual interpolation.
- Pure statistical Bayesian approaches often have difficulty in faithfully representing and recovering image geometries.
- Geometric image models explicitly express image geometries (e.g., the regularity of level sets and jump sets) in terms of energies.
- Geometric measure and free boundary theories are useful in understanding the behavior of our models.
- Our models are computationally realized by nonlinear geometric PDEs, the level-set method, and Γ – convergence approximations.
- Future efforts will be focused on: (a) integration (of different tools: wavelets/stochastic/PDEs); (b) high-level vision (feature & pattern analysis); (c) efficient algorithms (for the nonlinear high-order PDEs).

That is all, folks…

Thank you for your patience!



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